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On the spectrum of the linear transport operator in a semi-infinite medium

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Abstract. The linear mono-energetic Boltzmann equation with isotropic scattering is considered for a semi-infinite medium in plane geometry and the spectrum of the corresponding operator under perfectly reflecting, vacuum, generalized or diffusely reflecting boundary conditions is explored in the frame of the 'initial-value problem'. By the Hille–Yosida theorem, the existence and uniqueness of the solutions of these problems are assured. As a common feature, one observes the absence of a true isolated asymptotic eigenmode, the solution displaying, due to the infinite extent of the medium, only 'transient' modes.

1. Introduction

The semi-infinite medium problem in linear transport theory probably goes back to Milne in connection with diffusion and radiation in astrophysics. Since then, a considerable effort has been made to solve this problem and connected ones, using alternatively the Wiener–Hopf technique, the singular eigenfunction expansion and computational methods (Chandrasekhar 1960, Kuščer and Zweifel 1965, Zweifel 1967, Case and Zweifel 1967, Williams 1971).

Apart from the practical interest which it raises, it is very instructive to see in this problem one of the simplest natural realizations of an integro–differential equation, whose solutions are often not so hard to deduce or, at least, to approximate.

Let us outline the ideas: we are dealing with the fate of an initial distribution of neutrons $n_0(x, \mu)$ present at the time $t = 0$ in a scattering, non-absorbing, non-multiplicative medium filling the left half-space, whose evolution is governed by the linear mono-energetic Boltzmann equation with isotropic scattering:

$$\frac{\partial n}{\partial t}(x, \mu, t) = (An)(x, \mu, t) \tag{1}$$

where

$$A = -\mu \frac{\partial}{\partial x} - 1 + \frac{1}{2}\int_{-1}^{1} d\mu' \tag{2}$$

and all the other symbols have their usual meanings.

In order to be properly posed (and hopefully solvable), the Boltzmann equation (1) has to be supplemented with boundary conditions which guarantee the uniqueness of the solution in a certain function space.

Our aim is to show, in the frame of the functional analysis, that a number of boundary conditions (accounting more or less for the physical properties of the boundary) can be imposed on the Boltzmann operator (2) to convert it into a maximal dissipative operator or, in other words, into the infinitesimal generator of a proper evolution semi-group. Furthermore, we shall be able to find the spectrum of these dissipative operators, or at least to determine some characteristic part of it.

Let us denote by $\mathscr{H}$ the Hilbert space $L_2((-\infty, 0]\times[-1, 1])$ of the square integrable functions of x and μ, defined on $(-\infty, 0]\times[-1, 1]$ with the usual scalar product and norm.

The boundary conditions we are here concerned with are:

(i) Generalized boundary conditions:

$$n(0, -\mu) = \alpha(\mu)n(0, \mu)$$
$$0<\alpha(\mu)<1 \qquad \text{and} \qquad \alpha(-\mu)=\alpha^{-1}(\mu)$$

by definition. For $\alpha(\mu)=1$ or $\alpha(\mu)=0$ we obtain as particular cases the perfectly reflecting (and also the backward reflecting) and 'vacuum' (perfectly absorbing) boundary conditions respectively.

(ii) Diffusely reflecting boundary conditions:

$$n(0, \mu<0) = \int_0^1 n(0, \mu')\mathrm{d}\mu'.$$

So far, these conditions have been considered only for slab and bounded geometries (Lehner and Wing 1955, Belleni-Morante 1970, Angelescu *et al* 1974, 1975, 1976a, Protopopescu and Corciovei 1976) but none of them have been analysed for the time-dependent semi-infinite problem. The difference which will appear here, encountered also in the infinite medium problem (Beauwens and Mika 1969), will be the absence of true exponential decay modes.

Let us first consider the structure of the transport operator A; it contains the 'unperturbed' unbounded part $-\mu(\partial/\partial x)-1$ and the bounded perturbation $\frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu = J$. Under the imposed boundary conditions (each of (i) and (ii)), $-\mu(\partial/\partial x)-1$ turns out to be an operator satisfying the hypotheses of the Hille–Yosida theorem (Butzer and Berens 1967). As no misunderstanding is possible, let us denote the operator by $T-1$ in all cases and define it respectively on:

$\mathscr{D}(T-1)=\{n\in\mathscr{H}|n$ absolutely continuous in x, $\mu(\partial n/\partial x)\in\mathscr{H}$, n satisfies one of the conditions (i) or (ii)$\}$

by:

$$(T-1)n=-\mu\frac{\partial n}{\partial x}-n, \qquad n\in\mathscr{D}(T-1).$$

We have to prove:

(1) $\mathscr{D}(T-1)$ is dense in $\mathscr{H}$;
(2) $T-1$ is closed;
(3) $\|(\lambda-T+1)^{-1}\|\leq[1/(\lambda+1)]$, $\lambda>-1$.

Indeed:

(1) $\mathscr{D}(T-1)$ always contains the infinitely differentiable functions of x and μ with compact support in $(-\infty, 0]\times[-1, 1]$ which trivially satisfy all boundary conditions (i)–(ii) and which are dense in $\mathscr{H}$.

(2) The effective solving of the resolvent equation:

$$(\lambda - T+1)n = g, \qquad g \in \mathscr{H}, \tag{3}$$

shows in each case that the resolvent set $\rho(T-1)\neq\varnothing$. Hence for $\lambda\in\rho(T-1)$, $(\lambda - T+1)^{-1}$ exists as a bounded operator $R_\lambda(T-1)$ defined on $\mathscr{H}$, and therefore is closed. But, if S is a closed operator and its inverse S^{-1} exists, then S^{-1} is closed too (Kato 1966); thus T is closed. T is obviously dissipative, hence:

$$\|n\| \cdot \|g\| \geqslant |(n, g)| \geqslant |\mathrm{Re}(n, g)| \geqslant (\lambda+1)\|n\|^2 \tag{4}$$

and

$$\|(\lambda - T+1)^{-1}g\| = \|n\| \leqslant \frac{1}{\lambda+1}\|g\| \qquad \lambda > -1$$

that is

$$\|(\lambda - T+1)^{-1}\| = \|R_\lambda(T-1)\| \leqslant \frac{1}{\lambda+1} \qquad \lambda > -1. \tag{5}$$

Thus, by the Hille–Yosida theorem (Butzer and Berens 1967), $T-1$ generates a strongly continuous evolution semi-group $U_0(t)$ of contractive operators for $t>0$. Their concrete form shows that these operators are positivity preserving.

Since J is a bounded, positivity preserving perturbation, $T-1+J=A$ remains the infinitesimal generator of a strongly continuous semi-group of bounded, positivity preserving operators, $U(t)$, whose action can be deduced from the action of $U_0(t)$ by means of the perturbation formulae (Kato 1966):

$$\begin{aligned} U(t) &= \sum_{n=0}^{\infty} U_n(t) \\ U_n(t) &= \int_0^t U_0(t-\tau) J U_{n-1}(\tau)\, \mathrm{d}\tau. \end{aligned} \tag{6}$$

By these formulae, the existence and the uniqueness of the solution of the initial-value problems are ensured.

Unfortunately, the recurrence expression (6) is not very transparent and the features of the solution may remain rather obscure. A more obvious, although less complete method, consists in finding the spectrum of A and in deducing from it the behaviour of the solution. More precisely, in order to effectively write down the solution, one needs the spectral decomposition of the Boltzmann operator A. This is not at all a trivial requirement as A is not spectral even in the simplest cases. A less ambitious task would be the estimation of the asymptotic behaviours of the solutions in the manner initiated by Lehner and Wing (1956).

In fact, for slab and bounded geometries, the perturbation J, being relatively compact with respect to $T-1$, induces only some isolated eigenvalues of finite multiplicities; these eigenvalues are located to the right of the rest of the spectrum so their contribution will dominate the asymptotic behaviour. Moreover, by considering the equivalent integral equation one can show that the discrete eigenvalues are real for many interesting situations. Unfortunately, this reasoning cannot be followed here because J is not relatively compact with respect to $T-1$, theorem IV, (5.35) in Kato (1966) on the invariance of the essential spectrum does not apply and the spectrum induced by J must be otherwise analysed.

2. Generalized boundary conditions

Let us begin our study with the following problem: find the spectrum of the operator A with boundary conditions (i) (obviously related to a partial reflection of the neutrons at the boundary surface $x=0$, some part of the impinging distribution being absorbed).

We first assert:

Proposition 1. Under conditions (i) the unperturbed operator $T-1$ decomposes the spectral plane as follows:

(*a*) point spectrum: $\sigma_p(T-1)=\varnothing$;
(*b*) residual spectrum: $\sigma_r(T-1)=\varnothing$;
(*c*) continuous spectrum: $\sigma_c(T-1)=\{\lambda \mid \operatorname{Re}\lambda=-1\}$;
(*d*) resolvent set: $\rho(T-1)=\mathbb{C}\backslash\sigma_c(T-1)$.

Proof. By properly extending $n_0(x,\mu)$ with zero for $x>0$, the free evolution semi-group generated by $-\mu(\partial/\partial x)-1$ under conditions (i) gives:

$$(U_0(t)n_0)(x,\mu)=\mathrm{e}^{-t}(n_0(x-\mu t,\mu)+\alpha^{-1}(\mu)n_0(-x+\mu t,-\mu)). \tag{7}$$

Clearly,

$$\|U_0(t)n_0\|^2\leq\int_{-\infty}^{0}\mathrm{d}x\int_{-1}^{1}\mathrm{d}\mu\ \mathrm{e}^{-2t}(|n_0(x-\mu t)|^2+\alpha^{-2}|n_0(-x+\mu t)|^2)\leq \mathrm{e}^{-2t}\|n_0\|^2. \tag{8}$$

These estimations show that the abscissa of the free evolution semi-group is $\operatorname{Re}\lambda=-1$, i.e. the operator $T-1$ has the right half-plane $\{\lambda \mid \operatorname{Re}\lambda>-1\}$ as a resolvent set. A similar estimation for:

$$(U_0(-t)n_0)(x,\mu)=\mathrm{e}^{t}(n_0(x+\mu t,\mu)+\alpha^{-1}(\mu)n_0(-x-\mu t,-\mu)) \tag{9}$$

implies that $-T+1$ has the right half-plane $\{\lambda \mid \operatorname{Re}\lambda>1\}$ as a resolvent set and finally $\{\lambda \mid \operatorname{Re}\lambda<-1\}\subset\rho(T-1)$.

We see that although T is not a skew-adjoint operator ($\mathscr{D}(T)\neq\mathscr{D}(T^*)$) its (continuous) spectrum reduces only to the axis $\{\lambda \mid \operatorname{Re}\lambda=-1\}$. This can be proved by using the well known criterion for (continuous and discrete) spectra (Lehner and Wing 1955) with the function set:

$$f_\delta(x,\mu)=\begin{cases}\dfrac{1}{\delta}\mathrm{e}^{-\mathrm{i}\tau(x/\mu)}\,\mathrm{e}^{-(x^2/2)}\,\mathrm{e}^{-(1/2x^2)} & \delta^2\leq\mu\leq\delta,\quad \lambda=-1+\mathrm{i}\tau;\quad \delta\to 0\\ 0 & \text{otherwise.}\end{cases} \tag{10}$$

For eliminating the point spectrum, let us remark that any square integrable solution of the eigenvalue equation $(\lambda+\mu(\partial/\partial x)+1)n=0$ cannot satisfy the boundary conditions (i).

Finally, let $\lambda \in \sigma_r(T-1)$, which implies $((\lambda - T+1)\mathscr{D}(T))^{\perp} \neq \{0\}$. As the domain of T^*:

$\mathscr{D}(T^*) = \{n \in \mathscr{H} | n$ absolutely continuous in x,

$$\mu(\partial n/\partial x) \in \mathscr{H}, n(0, \mu) = \alpha(\mu) n(0, -\mu), \mu \in (0, 1]\}$$

is dense in $\mathscr{H}$, there exists $g \neq 0, g \in ((\lambda - T+1)\mathscr{D}(T))^{\perp} \cap \mathscr{D}(T^*)$. For this g and any $n \in \mathscr{D}(T)$, $((\lambda - T+1)n, g) = 0 = (n, (\bar{\lambda} - T^* + 1)g)$. As $\mathscr{D}(A)$ is dense in $\mathscr{H}$, $(\bar{\lambda} - T^* + 1)g = 0$ implies $\bar{\lambda} \in \sigma_p(T^*-1)$. From the domain condition and the reality of the operator A, we have $\sigma_p(T^*-1) = \sigma_p(T-1) = \overline{\sigma_p(T-1)}$ and $\bar{\lambda} \in \sigma_p(T-1) \Rightarrow \lambda \in \sigma_p(T-1)$ in contradiction to the initial assumption.

Let us now return to the perturbed transport operator (2) under a particular form of conditions (i).

Proposition 2. For $\alpha(\mu) = \alpha =$ constant, the spectrum of A is:
(*a*) $\sigma_r(A) = \varnothing$;
(*b*) $\sigma_p(A) \cup \sigma_c(A) = \{\lambda \,|\, \mathrm{Re}\,\lambda = -1\} \cup \{\lambda \in \mathbb{R} | -1 < \lambda \leq 0\}$;
(*c*) $\rho(A) = \mathbb{C} \backslash (\sigma_p(A) \cup \sigma_c(A))$.

Proof. (*a*) The residual spectrum is empty as before. From equation (7) the resolvent of $T-1$ is derived by Laplace transform for $\mathrm{Re}\,\lambda \neq -1$:

$$(R_\lambda(T-1)n_0)(x, \mu) = \begin{cases} \dfrac{1}{\mu}\displaystyle\int_{-\infty}^{x} n_0(x', \mu)\, \mathrm{e}^{-[(\lambda+1)/\mu](x-x')}\, \mathrm{d}x'; & \dfrac{\mathrm{Re}\,\lambda+1}{\mu} > 0 \\ -\dfrac{1}{\mu}\displaystyle\int_{x}^{0} n_0(x', \mu)\, \mathrm{e}^{-[(\lambda+1)/\mu](x-x')}\, \mathrm{d}x' & \\ -\dfrac{1}{\alpha(\mu)}\dfrac{1}{\mu}\displaystyle\int_{-\infty}^{0} n_0(x', -\mu)\, \mathrm{e}^{-[(\lambda+1)/\mu](x+x')}\, \mathrm{d}x'; & \dfrac{\mathrm{Re}\,\lambda+1}{\mu} < 0. \end{cases} \tag{11}$$

For $\mathrm{Re}\,\lambda > -1$ we prefer to study the equivalent integral equation $\phi = JR_\lambda(T-1)J\phi$, $\phi = Jn \in L_2((-\infty, 0])$ instead of the eigenvalue equation $(\lambda - A)n = 0$. If for some $\lambda \in \{\lambda \,|\, \mathrm{Re}\,\lambda > -1\}$, $1 \in \rho(JR_\lambda J)$ then $\lambda \in \rho(A)$ and vice versa (Beauwens and Mika 1969).

From equation (11), the equivalent integral operator is obtained:

$(JR_\lambda(T-1)J)(x, x')$

$$\stackrel{\mathrm{def}}{=} H_\lambda(x, x') = \frac{1}{2}\int_1^\infty \frac{\mathrm{d}t}{t}(\mathrm{e}^{-(\lambda+1)t|x-x'|} + \alpha\, \mathrm{e}^{-(\lambda+1)t|x+x'|})$$

$$= (1-\alpha)\frac{1}{2}\int_1^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-(\lambda+1)t|x-x'|} + \frac{\alpha}{2}\int_1^\infty \frac{\mathrm{d}t}{t}(\mathrm{e}^{-(\lambda+1)t|x-x'|} + \mathrm{e}^{-(\lambda+1)t|x+x'|})$$

$$\stackrel{\mathrm{def}}{=} (1-\alpha)H_\lambda^{(1)} + \alpha H_\lambda^{(2)}. \tag{12}$$

We shall show that for $\mathrm{Re}\,\lambda > -1$, $\mathrm{Im}\,\lambda \neq 0$, H_λ cannot have 1 as an eigenvalue and, consequently, λ cannot be an eigenvalue of A.

Indeed, let us suppose that $\phi = H_\lambda \phi$. Then,

$$0 \leqslant (\phi, \phi) = (H_\lambda \phi, \phi) = (1-\alpha)(H_\lambda^{(1)}\phi, \phi) + \alpha(H_\lambda^{(2)}\phi, \phi). \tag{13}$$

The scalar products appearing in the right-hand side of equation (13) can be computed in the Fourier representation. The Fourier transforms of $H_\lambda^{(1)}(x, x')$, $H_\lambda^{(2)}(x, x')$ are calculated for $\lambda > -1$, but are valid by analytic continuation also for $\mathrm{Re}\,\lambda > -1$:

$$H_\lambda^{(1)}(x - x') = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{H}_\lambda^{(1)}(k)\,\mathrm{e}^{-\mathrm{i}k(x-x')}\,\mathrm{d}k \tag{14}$$

$$\tilde{H}_\lambda^{(1)}(k) = \frac{1}{k}\tan^{-1}\frac{k}{\lambda+1}. \tag{15}$$

In order to compute the Fourier transform of $H_\lambda^{(2)}(x, x')$ we note that $H_\lambda^{(2)}(x, x')$ is an even function of x and x' separately and

$$\tilde{H}_\lambda^{(2)}(k) = \frac{1}{k}\tan^{-1}\frac{k}{\lambda+1}. \tag{16}$$

Thus (13) becomes

$$0 \leqslant (1-\alpha)\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{H}_\lambda^{(1)}(k)|\tilde{\phi}^{(1)}(k)|^2\,\mathrm{d}k + \alpha\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{H}_\lambda^{(2)}(k)|\tilde{\phi}^{(2)}(k)|^2\,\mathrm{d}k \tag{17}$$

where

$$\begin{aligned} \tilde{\phi}^{(1)}(k) &= \int_{-\infty}^{0} \phi(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \\ \tilde{\phi}^{(2)}(k) &= \int_{-\infty}^{0} \phi(x)\cos kx\,\mathrm{d}x. \end{aligned} \tag{18}$$

But, as for $\mathrm{Im}\,\lambda \gtrless 0$, we have $\mathrm{Im}\,(1/k)\tan^{-1}[k/(\lambda+1)] \lessgtr 0$, the reality of the spectrum of A is ensured for $\mathrm{Re}\,\lambda > -1$.

The axis $\{\lambda\,|\,\mathrm{Re}\,\lambda = -1\}$ and the real segment $\{\lambda \in \mathbb{R}|-1 < \lambda \leqslant 0\}$ are covered by spectra (continuous and eventually—although improbably—discrete) by using respectively the set (10) and the set:

$$f_T(x, \mu) = \sqrt{\frac{1}{2(\lambda+1)}}\,\frac{1}{1+\mathrm{i}x_0\mu}\,\mathrm{e}^{\mathrm{i}x_0(\lambda+1)x}\,\psi_T(x) \tag{19}$$

where x_0 is the real solution of the equation $[1/x_0(\lambda+1)]\tan^{-1} x_0 = 1$, $\lambda \in (-1, 0]$ and

$$\psi_T(x) = \begin{cases} \dfrac{1}{T^{1/2}}\mathrm{e}^{(x+T)} & x < -T; \qquad T \to \infty \\ -\dfrac{x}{T^{3/2}} & -T \leqslant x \leqslant 0. \end{cases} \tag{20}$$

Now, by considering the actual form of the unperturbed resolvent operator for $\mathrm{Re}\,\lambda < -1$ and constructing the corresponding equivalent integral equation, it can easily be seen that $\{\lambda\,|\,\mathrm{Re}\,\lambda < -1\} \subset \rho(A)$. Finally, for $\lambda \in \{\lambda\,|\,\mathrm{Re}\,\lambda > 0\}$, $(\lambda - A)^{-1}$ is bounded and thus the proof is achieved.

For $\alpha(\mu)=1$ (perfectly reflecting boundary conditions) the results can be strengthened in the sense that $\sigma_p(A)=\varnothing$.

Indeed, for $\alpha(\mu)=\alpha=1$, the convolution equation (13) is equivalent to

$$\tilde{\phi}(k)=\frac{1}{k}\tan^{-1}\frac{k}{\lambda+1}\tilde{\phi}(k). \tag{18'}$$

But, for $\lambda\in(-1,0]$, $(1/k)\tan^{-1}[k/(\lambda+1)]$ takes the value 1 on a set of measure zero; thus the set $\{\lambda\in\mathbb{R}|-1<\lambda\leqslant 0\}$ does not contain a point spectrum.

Now let λ be on the axis $\{\lambda|\operatorname{Re}\lambda=-1\}$. If $-1+\mathrm{i}\tau$ is an eigenvalue of A then $\operatorname{Re}((\lambda-A)n,n)=0$ implies $\|Jn\|^2=0$ because $\operatorname{Re}(Tn,n)=0$. But $Jn=0$ leads to $-1+\mathrm{i}\tau\in\sigma_p(T-1)$, which is impossible.

3. Vacuum boundary conditions

3.1. Case 1

For $\alpha(\mu)=0$, the conditions (i) describe the evolution in a semi-infinite medium with perfectly absorbing boundary. We prefer to study this case separately because some interesting features can be pointed out.

Proposition 3. The spectrum of the Boltzmann operator (2) under the boundary condition $n(0,\mu<0)=0$ is:

(*a*) $\sigma_r(A)=\varnothing$;

(*b*) $\sigma_p(A)\cup\sigma_c(A)=\{\lambda|\operatorname{Re}\lambda\leqslant -1\}\cup\{\lambda\in\mathbb{R}|-1<\lambda\leqslant 0\}$;

(*c*) $\rho(A)=\mathbb{C}\backslash(\sigma_p(A)\cup\sigma_c(A))$.

Proof. The left half-plane $\{\lambda|\operatorname{Re}\lambda\leqslant -1\}$ belongs to the spectrum of A as can be seen by considering the function set:

$$\phi_\delta(x,\mu)=\begin{cases}\dfrac{1}{\delta}\mathrm{e}^{-[(\lambda+1)/\mu]x} & \delta^2\leqslant\mu\leqslant\delta; \qquad \delta\to 0\\[2ex] 0 & \text{otherwise.}\end{cases} \tag{21}$$

The rest of the proof is the same as in the previous case. The discrete eigenvalues on $(-1,0]$ are eliminated by using the results of the subsequent paragraph.

We remark that the filling up of the left half-plane with a continuous spectrum is discontinuous with respect to α and we shall give a possible explanation of this fact in § 4.

3.2. Case 2

For vacuum boundary conditions the fact that T is not skew-adjoint appears strongly because its spectrum fully covers a certain left half-plane. This is, surely, a disappointing feature which drastically removes the hope that A can ever allow for a spectral decomposition. This fact is not so unfortunate for the slab because of the existence of some isolated eigenvalues in the right part of the spectrum. A Riesz integral can be performed on this part, displaying at least an exponential asymptotic behaviour of the distribution. A slightly modified point of view introduced by Lehner (1962) in connection with the slab problem, can greatly simplify the spectrum. Indeed, let us consider the following problem instead of the one treated in § 3.1.

Find the spectrum of the transport operator considered as an operator in $L_2((-\infty,\infty)\times[-1,1])$ and defined by:

$$A_1=-\mu\frac{\partial}{\partial x}-1+\chi_{(-\infty,0]}(x)\frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu \tag{22}$$

where $\chi_{(-\infty,0]}(x)$ is the characteristic function of the semi-axis $(-\infty,0]$:

$$\chi_{(-\infty,0]}(x)=\begin{cases}1 & x\in(-\infty,0]\\ 0 & \text{otherwise.}\end{cases} \tag{23}$$

A_1 has the same action as A in the moderator while in the vacuum space the perturbation is dropped. Although it is expected (and proved) that the two operators give the same evolution (physically, the 'facts' described by (1) with vacuum boundary conditions and by (22) are the same inside the moderator), their spectra differ. We assert:

Proposition 4. The spectrum of A_1 is:
(*a*) $\sigma_r(A_1)=\varnothing$;
(*b*) $\sigma_p(A_1)=\varnothing$;
(*c*) $\sigma_c(A_1)=\{\lambda\,|\operatorname{Re}\lambda=-1\}\cup\{\lambda\in\mathbb{R}|-1<\lambda\leqslant 0\}$;
(*d*) $\rho(A_1)=\mathbb{C}\backslash\sigma_c(A_1)$.

Proof. As defined on $L_2((-\infty,\infty)\times[-1,1])$, $-\mu(\partial/\partial x)$ becomes a skew-adjoint operator T and the spectrum of $T-1$ is $\{\lambda\,|\operatorname{Re}\lambda=-1\}$ (see the set (10)).

The rest of the proof follows mainly as in the preceding section, the only difference being that the resolvent of $T-1$ is:

$$(R_\lambda(T-1)n_0)(x,\mu)=\begin{cases}\dfrac{1}{\mu}\displaystyle\int_{-\infty}^{x}n_0(x',\mu)\,\mathrm{e}^{-[(\lambda+1)/\mu](x-x')}\,\mathrm{d}x'; & \dfrac{\operatorname{Re}\lambda+1}{\mu}>0\\ -\dfrac{1}{\mu}\displaystyle\int_{x}^{\infty}n_0(x',\mu)\,\mathrm{e}^{-[(\lambda+1)/\mu](x-x')}\,\mathrm{d}x'; & \dfrac{\operatorname{Re}\lambda+1}{\mu}<0\end{cases} \tag{24}$$

and consequently

$$H_\lambda(x,x')=\frac{1}{2}\int_1^\infty\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-(\lambda+1)t|x-x'|}.$$

For eliminating the point spectrum on the real segment $(-1,0]$, one has to observe that the distributions being considered are in $L_2(-\infty,\infty)$; the equivalent integral equation can be written:

$$\phi(x)=\int_{-\infty}^{\infty}H_\lambda(x,x')\phi(x')\,\mathrm{d}x' \tag{25}$$

and we can naturally apply the Fourier transform obtaining

$$\tilde{\phi}(k)=\frac{1}{k}\tan^{-1}\frac{k}{\lambda+1}\tilde{\phi}(k). \tag{26}$$

Here the same argument is used as in § 2.

This also furnishes a proof that the discrete spectrum is eliminated on the real segment $(-1,0]$ even for vacuum conditions. Let us suppose to the contrary: if $\lambda\in(-1,0]$ is an eigenvalue of A with corresponding eigenfunction n, by properly extending this function for $[0,\infty)$ with

$$n'=\begin{cases}0 & x<0\\ n(0,\mu,t)\,\mathrm{e}^{-[(\lambda+1)/\mu]x} & x>0,\end{cases} \tag{27}$$

one would find that

$$\mathcal{N}=\begin{cases}n & x<0\\ n' & x>0\end{cases}$$

is an eigenfunction of A_1 corresponding to the same eigenvalue, which is impossible because A_1 has no eigenvalues on the real segment $(-1,0]$. We can use the same argument as that used for perfectly reflecting boundary conditions to prove that $\{\lambda\,|\,\mathrm{Re}\,\lambda=-1\}$ does not contain a point spectrum of A_1 and therefore of A.

3.3. Case 3

This 'simplified' problem suggests a more natural way of writing the evolution equation. Indeed, the operator A_1 describes a 'normal' evolution inside the moderator and a 'free damped' evolution outside. T is responsible for the 'free' part and in fact represents the infinitesimal generator of the translations in the x direction with velocity μ. On the other hand, the minus unity operator retained by Lehner in his original approach for the slab, is responsible for the 'damped' part. But if the right half-space is a vacuum, it cannot trap particles and, generally speaking, there is no material which would be characterized only by an 'in-part' of the scattering cross section and not by an 'out-part' too.

For such a medium the detailed balance principle (in a very rudimentary form!) would not hold. The only physical interpretation for -1 would be that it represents an absorption cross section equal to the total scattering cross section of the moderator. If, however, the right half-space is a vacuum, one is led to consider in $L_2((-\infty,\infty)\times[-1,1])$ the evolution equation:

$$\begin{gathered}\frac{\partial n}{\partial t}(x,\mu,t)=(A_2 n)(x,\mu,t)\\ A_2=-\mu\frac{\partial}{\partial x}-\chi_{(-\infty,0]}(x)\left(1-\frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\right)\end{gathered} \tag{28}$$

with $\chi_{(-\infty,0]}(x)$ defined by equation (23). This problem, although academic, displays some very important features which can appear in solving linear evolution equations (and, particularly, linear transport equations) and which are worth pointing out.

As $-\mu(\partial/\partial x)$, considered on infinitely differentiable functions with compact support in x, has a unique skew-adjoint extension T in $L_2((-\infty,\infty)\times[-1,1])$, we have $\mathrm{Re}(Tn,n)=0$ and the obvious estimations $\|(\lambda-A)n\|\geqslant-(\mathrm{Re}\,\lambda+1)\|n\|$ and

$\|(\lambda - A)n\| \geqslant \mathrm{Re}\,\lambda \|n\|$. Now, we shall prove that $\{\lambda \mid \mathrm{Re}\,\lambda = -1\} \cup \{\lambda \mid \mathrm{Re}\,\lambda = 0\} \cup \{\lambda \in \mathbb{R} \mid -1 < \lambda \leqslant 0\} \subset \sigma_c(A_2)$. Indeed, let us use respectively the function sets

$$f_\delta^{(1)}(x,\mu) = \begin{cases} \frac{1}{\delta} \mathrm{e}^{-\mathrm{i}\tau(x/\mu)} \mathrm{e}^{-(x^2/2)} \mathrm{e}^{-(1/2x^2)} & x \leqslant 0; \quad \lambda = -1 + \mathrm{i}\tau; \quad \delta^2 \leqslant \mu \leqslant \delta; \quad \delta \to 0 \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

$$f_\delta^{(2)}(x,\mu) = \begin{cases} \frac{1}{\delta} \mathrm{e}^{-\mathrm{i}\tau(x/\mu)} \mathrm{e}^{-(x^2/2)} \mathrm{e}^{-(1/2x^2)} & x \geqslant 0; \quad \lambda = \mathrm{i}\tau; \quad \delta^2 \leqslant \mu \leqslant \delta; \quad \delta \to 0 \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

and the set (15). The verification is straightforward and the result striking: the Boltzmann operator A_2 has, as continuous spectrum, the real segment $[-1, 0]$ and two infinite lines which enclose between them a whole strip of the spectral plane. This strip cannot be analysed, at least by standard methods, as it is completely bordered by spectral lines.

The key point for pushing forward the analysis is the remark that the spectrum in itself may contain redundant information and full knowledge of it is not always necessary (nor sufficient) for solving the problem.

In fact, for initial distributions restricted to the left half-space (the only ones with physical significance for our problem), the free evolution semi-group with vacuum boundary conditions gives:

$$(U_0(t)n_0)(x,\mu) = \mathrm{e}^{-t} n_0(x - \mu t, \mu) \qquad x, x - \mu t \in (-\infty, 0] \tag{31}$$

while in the Lehner approach:

$$(U_0^{(1)}(t)n_0)(x,\mu) = \mathrm{e}^{-t} n_0(x - \mu t, \mu) \qquad x - \mu t \in (-\infty, 0]. \tag{32}$$

In the present case, using the usual definitions of the characteristic functions $\chi_{(0,1]}(\mu)$, $\chi_{[-1,0)}(\mu)$, $\chi_{[0,\infty)}(x)$, the evolution semi-group is:

$$\begin{aligned}(U_0^{(2)}(t)n_0)(x,\mu) &= \{\chi_{(0,1]}(\mu)[\chi_{(-\infty,0]}(x)\,\mathrm{e}^{-t} + \chi_{[0,\infty)}(x)\,\mathrm{e}^{[(-x+\mu t)/\mu]}] \\ &\quad + \chi_{[-1,0)}(\mu)\chi_{(-\infty,0]}(x)\,\mathrm{e}^{-t}\} n_0(x - \mu t, \mu)\end{aligned} \tag{33}$$

from which the resolvent:

$$\begin{aligned}(R_\lambda(T - \chi_{(-\infty,0]})n_0)(x,\mu) &= \chi_{(0,1]}(\mu) \int_{-\infty}^{x} \frac{\mathrm{d}x'}{\mu} [\chi_{(-\infty,0]}(x)\chi_{(-\infty,0]}(x') + \chi_{[0,\infty)}(x)\chi_{[-\infty,0]}(x')\,\mathrm{e}^{(x/\mu)}] \\ &\quad \times \mathrm{e}^{-[(\lambda+1)/\mu](x-x')} n_0(x',\mu) - \chi_{[-1,0)}(\mu) \int_x^0 \frac{\mathrm{d}x'}{\mu} \chi_{(-\infty,0]}(x)\chi_{(-\infty,0]}(x') \\ &\quad \times \mathrm{e}^{-[(\lambda+1)/\mu](x-x')} n_0(x',\mu)\end{aligned} \tag{34}$$

is evidently bounded for $\mathrm{Re}\,\lambda > -1$ as before; i.e. by considering the same physical problem (n_0 with support contained in $(-\infty, 0] \times [-1, 1]$), the spurious part of the spectrum disappears.

By projecting the semi-group (33) only on the left half-space one obtains exactly the semi-group $U_0(t)$:

$$\chi_{(-\infty,0]}(x)(U_0^{(2)}(t)n_0)(x,\mu)$$
$$= \mathrm{e}^{-t}n_0(x-\mu t,\mu) = (U_0(t)n_0)(x,\mu) \qquad x, x-\mu t \in (-\infty, 0]. \tag{35}$$

The real, perturbed semi-group has the same property on account of the commutativity relations:

$$[\chi_{(-\infty,0]}(x), J] = [U_0^{(2)}(t), \chi_{(-\infty,0]}(x)] = 0. \tag{36}$$

With this reduction, the analysis can be carried out as in the previous case, concluding that in the moderator all the three semi-groups considered in §§§ 3.1, 3.2 and 3.3 give identical solutions. This makes the determination of the spectrum of A_2 superfluous and qualitatively relates the presence of the spurious part of the spectrum to the distributions evolving only in the right half-space at all t (this assertion is not quite exact).

4. Diffusely reflecting boundary conditions

As in the generalized case with $\alpha(\mu) \neq$ constant, the results are not as stringent as for the previous problems; however, in spite of their poorness, these results display an interesting and instructive feature, specifically, the presence of the spectrum in the whole left half-plane, as for the vacuum conditions.

Proposition 5. For the operator (2) under conditions (ii),

$$\{\lambda \mid \mathrm{Re}\,\lambda \leqslant -1\} \cup \{\lambda \in \mathbb{R} \mid -1 < \lambda \leqslant 0\} \subset \sigma(A) \text{ and } \{\lambda \mid \mathrm{Re}\,\lambda > 0\} \subset \rho(A).$$

Proof. The free evolution semi-group gives:

$$(U_0(t)n_0)(x,\mu) = n_0(x-\mu t,\mu) + \chi_{[-1,0)}(\mu)\int_0^1 n_0\left(-\mu'\left(t-\frac{x}{\mu}\right),\mu'\right)\mathrm{d}\mu' \tag{37}$$

which for $\mathrm{Re}\,\lambda > -1$ leads to the unperturbed resolvent:

$$(R_\lambda(T-1)n_0)(x,\mu) = \begin{cases} \dfrac{1}{\mu}\displaystyle\int_{-\infty}^{x} n_0(x',\mu)\,\mathrm{e}^{-[(\lambda+1)/\mu](x-x')}\,\mathrm{d}x' & \mu > 0 \\ -\dfrac{1}{\mu}\displaystyle\int_x^0 n_0(x',\mu)\,\mathrm{e}^{-[(\lambda+1)/\mu](x-x')}\,\mathrm{d}x' & \\ \quad + \displaystyle\int_0^1 \frac{\mathrm{d}\mu'}{\mu'}\int_{-\infty}^0 n_0(x',\mu')\,\mathrm{e}^{-(\lambda+1)[(x/\mu)-(x'/\mu')]}\,\mathrm{d}x' & \mu < 0. \end{cases} \tag{38}$$

The inclusion $\{\lambda \mid \mathrm{Re}\,\lambda > 0\} \subset \rho(A)$ follows as before.

Moreover, the set (19) can be used again to prove the inclusion $\{\lambda \in \mathbb{R} \mid -1 < \lambda \leqslant 0\} \subset \sigma(A)$, while for the left half-plane we use the set ($\lambda = \beta + \mathrm{i}\tau, \beta < -1$):

$$h_\delta(x,\mu) = \begin{cases} \dfrac{1}{\delta}\,\mathrm{e}^{-[(\lambda+1)/\mu]x}\sin\dfrac{2\pi\mu}{\delta} & 0 < \mu \leqslant \delta; \qquad \delta \to 0 \\ 0 & \text{otherwise} \end{cases} \tag{39}$$

which obviously satisfy (ii) and for which we have the estimations:

$$\|h_\delta\|^2 = \frac{1}{\delta^2}\int_0^\delta \int_{-\infty}^0 \mathrm{e}^{-[2(\beta+1)/\mu]x} \sin^2\frac{2\pi\mu}{\delta}\,\mathrm{d}x\,\mathrm{d}\mu$$

$$= -\frac{1}{\delta^2}\int_0^\delta \frac{\mu\,\mathrm{d}\mu}{2(\beta+1)}\frac{1-\cos(4\pi\mu/\delta)}{2} = -\frac{1}{8(\beta+1)} > 0. \tag{40}$$

$$\|(\lambda - A)h_\delta\|^2$$

$$= \frac{1}{4\delta^2}\int_{-\infty}^0 \mathrm{d}x \int_{-1}^1 \mathrm{d}\mu \left| \int_0^\delta \mathrm{e}^{-[(\lambda+1)/\mu']x} \sin\frac{2\pi\mu'}{\delta}\,\mathrm{d}\mu' \right|^2$$

$$\leqslant \frac{1}{4\delta^2}\int_{-\infty}^0 \mathrm{d}x \int_{-1}^1 \mathrm{d}\mu \left(\int_0^\delta \mathrm{e}^{-[(\beta+1)/\mu']x} \left|\sin\frac{2\pi\mu'}{\delta}\right|^2 \mathrm{d}\mu' \right)^2$$

$$\leqslant \frac{1}{4\delta^2}\int_{-\infty}^0 \mathrm{d}x \int_{-1}^1 \mathrm{e}^{-[2(\beta+1)/\delta]x}\,\delta^2\,\mathrm{d}\mu = -\frac{1}{4(\beta+1)}\delta \to 0 \quad \text{as} \quad \delta \to 0. \tag{41}$$

Why this apparently unexpected presence of the spectrum in the whole left half-plane? For vacuum conditions this is related to those initial distributions located very close to the boundary surface $x = 0$ which leave the medium, no matter how small the time interval. This 'disappearance' of the distributions prints an irreversible character on the evolution; this is governed by a *semi-group*, contrary to the case of reflecting or generalized boundary conditions, or of the Lehner treatment, when the evolution is governed by a *group*. The free unperturbed group generated by T, is unitary for the perfectly reflecting boundary conditions and for the Lehner case and non-unitary for the generalized boundary conditions, but in any case a reversed evolution can be completely determined.

The diffusely reflecting boundary conditions have, in common with the vacuum ones, the loss of 'memory' of the system at the surface $x = 0$. The situation is the same, independent of whether the neutron distribution leaves the system or is returned to it as a diffusely reflected packet; the original distribution cannot be reconstructed by changing the sign of the time (although the diffusely reflecting boundary conditions preserve the number of particles—no particle can escape from the medium) and this is the reason for the appearance of the spectrum in the whole left half-plane.

5. Conclusions

We have studied the spectrum of the Boltzmann operator (2) with several boundary conditions appropriately accounting for physical properties of the boundary surface of the medium. Some of them can actually appear (single or mixed) in the transport theory of neutrons, others can perhaps be of some importance for the transport of other particles or light.

Having proved in general the existence and uniqueness of the solution of the initial-value problem for a large variety of boundary conditions, the knowledge of the spectrum could allow, if not for a proper spectral decomposition of the operator and thus the effective solution, at least for the finding of its asymptotic behaviour.

The common feature of the initial-value problem, proved for many cases and conjectured for the others, is that in semi-infinite media the distributions behave essentially as in the infinite medium (Beauwens and Mika 1969) displaying only transient modes and no true asymptotic exponential modes.

The presence or absence of the continuous spectrum in the left half-plane $\{\lambda \mid \mathrm{Re}\,\lambda \leq -1\}$ is related to the 'memory-loss' of the system at the boundary surface $x = 0$.

Finally, the possibility of conveniently including these transients in the solution of the time-dependent problem is not trivial and is related to the possibility of some generalized spectral decomposition of the Boltzmann operator which holds in a sense only if the closure of $-\mu(\partial/\partial x)$ is a skew-adjoint operator (Angelescu *et al* 1976b).

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